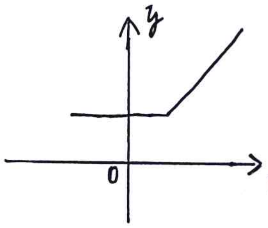
Tutorial notes - 2   Deng Yongzhe \ whizkiddeng@gmail.com

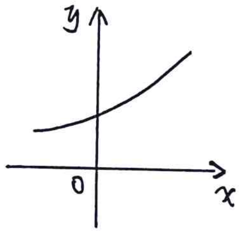
The properties of function — Monotonicity, periodicity and symmetry.

1. Monotonicity — increasing and decreasing

From the graph:



increasing or non-decreasing



strictly increasing

Definition:

$\forall\, x_1, x_2 \in D(f)$, $x_1 < x_2$

then $f(x_1) \le f(x_2)$ (increasing or non-decreasing)

For the strictly increasing, we have to take off the "=", means:

$\forall\, x_1, x_2 \in D(f)$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

similarly we can get the decreasing part.

Remark: Monotonicity is very important when we talk about the injective and inverse later.

2. Periodicity.

Def: There exists a positive number(T), for $\forall x \in D(f)$, we have:

$f(x+T) = f(x)$ (so we must make sure $x+T \in D(f)$ too).

then T is called period of f.

Remark: For $f(x) = f(x+T) = f(x+2T) = \cdots$ so $2T$ also is period of f, but we just concern the smallest one, means T, for the other periods just come from it.

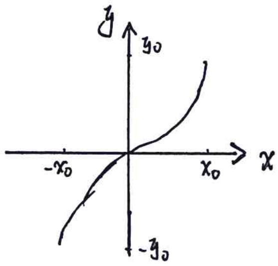
3. Symmetry.

The most common case for symmetry are odd / even function.

odd function:

Def: for $\forall x \in D(f)$, we have: $f(x) = -f(-x)$ (so $f(0)=0$)

Graph:



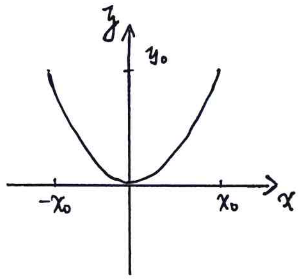
symmetric to $(0,0)$

"point symmetry"

even function:

Def: $\forall x \in D(f) \Rightarrow f(x) = f(-x)$

Graph:



symmetric to $x=0$ (y-axis)

" line symmetry "

Remark: From the definiton we can see: for odd \ even function, $x \in D(f) \Rightarrow -x \in D(f)$

which means its domain $D(f)$ should be symmetric, like:

$[-5, 5]$, $(-3, 3)$, or $(-3,0) \cup (0,3)$ all symmetric.

2. useful formulas:

(1) If $f(x)$ is symmetric to $x = a$, then $f(x) = f(2a-x)$

(2) If $f(x)$ is symmetric to point (a, b), then $f(x) = 2b - f(2a-x)$

Q1. Discuss the 3 properties of following functions.

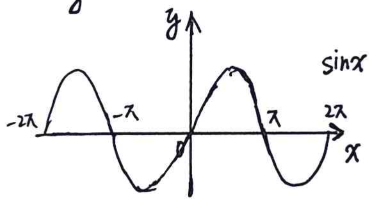
(1) $y = e^x$ $D(f) = R$

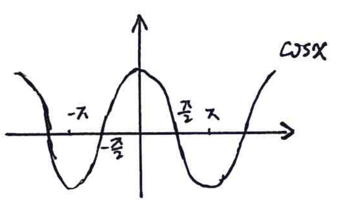
Generally for $y = a^x$ $(a \neq 0, a > 0)$ = $\begin{cases} \text{strictly increasing} & a > 1 \quad (e = 2.71.. > 1) \\ \text{constant} & a = 1 \\ \text{strictly increasing} & 0 < a < 1 \end{cases}$

for the $\{a>1\}$, $\{0<a<1\}$ cases. we don't have any "equality" which imples

$y = a^x$ can't be period and symmetric.

(2) $y = \sin x$, $\cos x$, $\tan x$.





The First thing we should do is to

check the domain of these functions.

If we use max-domain, everything is fine, for $D(\sin x) = D(\cos x) = R$.

then $\sin x$ is a odd function with period $T = 2\pi$

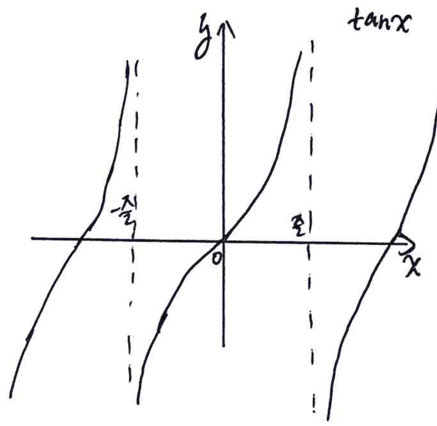
while $\cos x$ is a even function with period $T = 2\pi$

But we can't say they are increasing or decreasing for both cases exist

so we have to discuss independently.

But if we restrict the domain to $[0, \frac{\pi}{2}]$

then we don't have periodicity and symmetry, but we can $\begin{cases} \sin x \text{ is a increasing function} \\ \cos x \text{ is a decreasing function.} \end{cases}$



it's similar to discuss $\tan x$.

Q2.

(1) Assume $\frac{f(x)}{x}$ is decreasing in $(0,+\infty)$, show:

$$f(a+b) \le f(a) + f(b) \quad , \forall a,b \in (0,+\infty)$$

(2) Assume $\frac{f(x)}{x}$ is increasing in $(0,+\infty)$, show:

$$f(a+b) \ge f(a) + f(b) \quad , \forall a,b \in (0,+\infty)$$

Pf: (1) Try to construct the form of " $\frac{f(x)}{x}$ ", and we assume $a \ge b$.

$$\frac{f(a+b)}{a+b} \le \frac{f(a)}{a} \quad \left(\text{From } \frac{f(x)}{x} \downarrow\right) \qquad (1)$$

then we just have to show: $\frac{f(a)}{a} \le \frac{f(a)}{a+b} + \frac{f(b)}{a+b}$.

$$\Updownarrow$$

$$f(a)\left(\frac{1}{a} - \frac{1}{a+b}\right) = \frac{f(a)\,b}{a(a+b)} \le \frac{f(b)}{a+b}$$

$$\Updownarrow$$

$$\frac{f(a)}{a} \le \frac{f(b)}{b}.$$

And the last inequality is trival for we assume $a \ge b$, then $\frac{f(a)}{a} \le \frac{f(b)}{b}$ $\left(\frac{f(x)}{x} \downarrow\right)$

Remark: if $a<b$. we just choose $\frac{f(b)}{b}$ in (1) as intermediate value is ok.

(2) this part is the same with (1)

Q3. Assume $f(x)$ is symmetric to $x=a$ and $x=b$. $a>b$

Try to prove $f(x)$ is a period function, and compute its period.

Pf: Recall the formula of line symmetry. we have:

$f(x) = f(2a-x)$, $f(x) = f(2b-x)$

let $t = 2b-x$.

so $f(t) = f(2a-t) = f(2a-(2b-x)) = f(x+2a-2b)$

while $f(t) = f(2b-x) = f(x)$, so $f(x) = f(x+2a-2b)$

which means $T = 2a-2b > 0$.

Q4. Assume we have $f(x)$ defined in $[-l, l]$, try to show there must exist an odd function $g(x)$ and an even function $h(x)$ which also defined in $[-l,l]$ s.t $f(x) = g(x) + h(x)$.

Pf: Actually we just have to set:

$$g(x) = \frac{f(x)-f(-x)}{2} , \quad h(x) = \frac{f(x)+f(-x)}{2} .$$

then $f(x) = g(x) + h(x)$ is trival

And $g(-x) = \dfrac{f(-x)-f(x)}{2} = -g(x)$, $g(x)$ is odd.

$f(-x) = \dfrac{f(-x)+f(x)}{2} = f(x)$, $f(x)$ is even. Done.

But a question is: how can we think of such construction?

Actually we just need assume we have such represention first, means:

$f(x) = g(x) + h(x)$ (1) and $g(x)$ is odd, $h(x)$ is even.

then we know the odd and even both have relationship to $(-x)$, so we set x be $-x$:

$f(-x) = g(-x) + h(-x) = -g(x) + h(x)$ (2)

combine (1), (2); we can solve $g(x)$, $h(x)$ out:

just $g(x) = \dfrac{f(x)-f(-x)}{2}$, $h(x) = \dfrac{f(x)+f(-x)}{2}$

Math 1010 Tutorial 2 (Prepared by Chung Shun Wai)

Topics : Odd and even function , Injectivity and Surjectivity

Q1 : Determine whether the function is odd or even

i) $f : \mathbb{R}^{\times} \to \mathbb{R}^{+}$; $f(x) = x^{-2}$

ii) $f : \mathbb{R} \to \mathbb{R}$; $f(x) = \dfrac{e^{x} - e^{-x}}{2}$

Q2 : Determine whether the function is injective or surjective

i) $f : \mathbb{R} \to \mathbb{R}$; $f(x) = |x - 2| + 3$

ii) $f : \mathbb{R} \setminus \{2\} \to \mathbb{R}$; $f(x) = \dfrac{3x+1}{x-2}$

iii) $f : \mathbb{R}^{\times} \to \mathbb{R}^{+}$; $f(x) = x^{-2}$

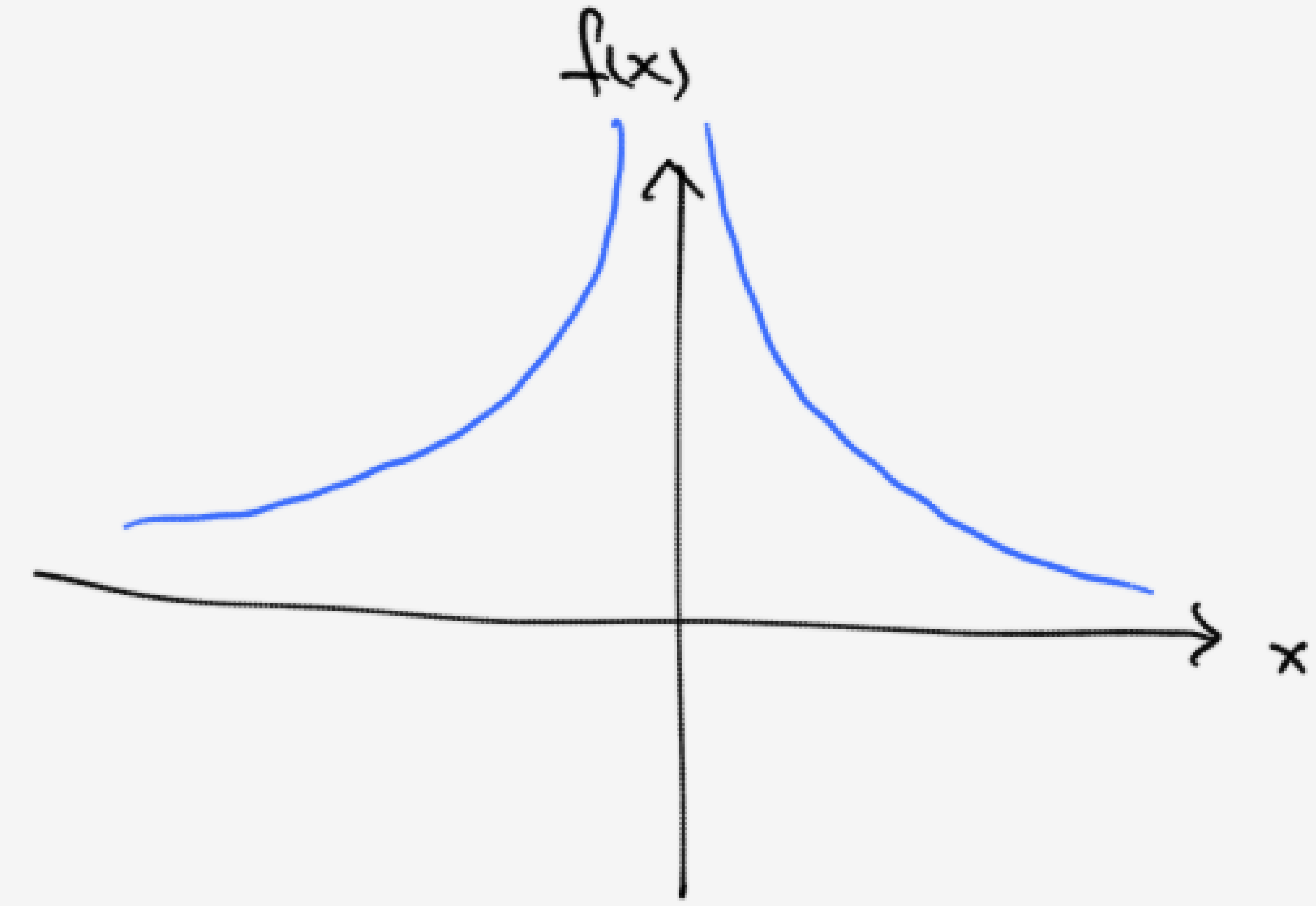
iv) $f : \mathbb{R} \to \mathbb{R}$; $f(x) = \dfrac{e^{x} - e^{-x}}{2}$

Sol^n

(i) given $f(x) = \frac{1}{x^2}$,

$$f(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = f(x)$$

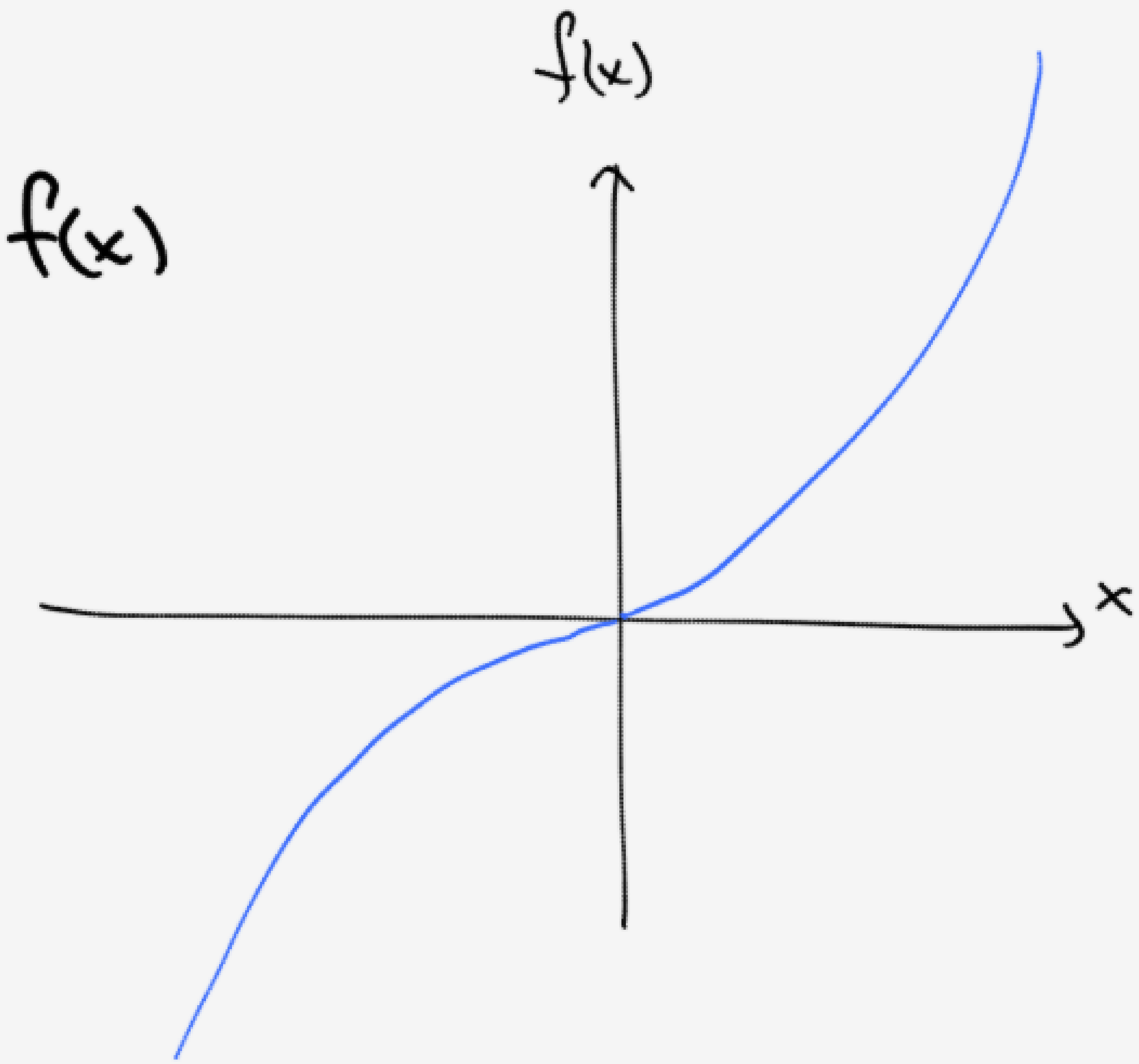
Hence f is even



(ii) given $f(x) = \frac{e^x - e^{-x}}{2}$

$$f(-x) = \frac{e^{-x} - e^x}{2} = -\frac{e^x - e^{-x}}{2} = -f(x)$$

Hence f is odd



2i) given $f : \mathbb{R} \to \mathbb{R}$; $f(x) = |x-2| + 3$

Inj: Suppose $f(x) = f(y) \Rightarrow |x-2| + 3 = |y-2| + 3$

$\Rightarrow x-2 = \pm(y-2) \Rightarrow x = y$ or $x = -y+4$

In particular, $x=1, y=3$ we have $f(1) = f(3) = 4$

Hence f is not inj.

Surj Notice that $\forall x \in \mathbb{R}$,

$f(x) = |x-2| + 3 \geq 3 \Rightarrow \text{Range}(f) \subseteq [3, \infty)$

Hence f is not surj.

2ii) Given $f: \mathbb{R}\setminus\{2\} \to \mathbb{R}$; $f(x) = \dfrac{3x+1}{x-2}$.

Inj : Suppose that $f(x) = f(y)$ for some $x, y \in \mathbb{R}\setminus\{2\}$.

$\Rightarrow \dfrac{3x-1}{x-2} = \dfrac{3y-1}{y-2} \Rightarrow (3x-1)(y-2) = (3y-1)(x-2)$

$\Rightarrow 3xy + 2 - y - 6x = 3xy + 2 - x - 6y \Rightarrow 7x = 7y \Rightarrow x = y$

Hence f is injective.

Surj. Notice that $3 \notin$ Range of f.

Assume the contrary that $f(x) = 3$ for some $x \in \mathbb{R}\setminus\{2\}$.

then $3 = \dfrac{3x+1}{x-2} \Rightarrow 3x - 6 = 3x + 1 \Rightarrow 7 = 0$ (impossible)

Hence f is not surjective.

2 iii) given $f: \mathbb{R}^{\times} \to \mathbb{R}^{+}$; $f(x) = \frac{1}{x^2}$

<u>Inj.</u> from question (1i), f is even. (i.e. $f(-x) = f(x)$)

in particular $f(-1) = f(1) = 1$

Hence f is not injective.

<u>Surj</u> Fix $y \in \mathbb{R}^{+}$, Suppose that $f(x) = y$ for some $x \in \mathbb{R}^{\times}$

then $f(x) = \frac{1}{x^2} = y \Rightarrow x^2 = \frac{1}{y} > 0 \Rightarrow x = \pm \frac{1}{\sqrt{y}}$

Hence f is surjective

2iv) given $f: \mathbb{R} \to \mathbb{R}$; $f(x) = \dfrac{e^x - e^{-x}}{2}$

Inj: Suppose that $f(x) = f(y) \Rightarrow \dfrac{e^x - e^{-x}}{2} = \dfrac{e^y - e^{-y}}{2}$

$\Rightarrow e^x - e^y + \dfrac{1}{e^y} - \dfrac{1}{e^x} = 0 \Rightarrow e^x - e^y + \dfrac{e^x - e^y}{e^{x+y}} = 0$

$\Rightarrow (e^x - e^y)(1 + \dfrac{1}{e^{x+y}}) = 0 \Rightarrow e^x = e^y \Rightarrow x = y$

Hence f is injective.

Surj: fix $y \in \mathbb{R} = \text{Codomain}(f)$ Suppose $f(x) = y$ $\exists x \in \mathbb{R}$

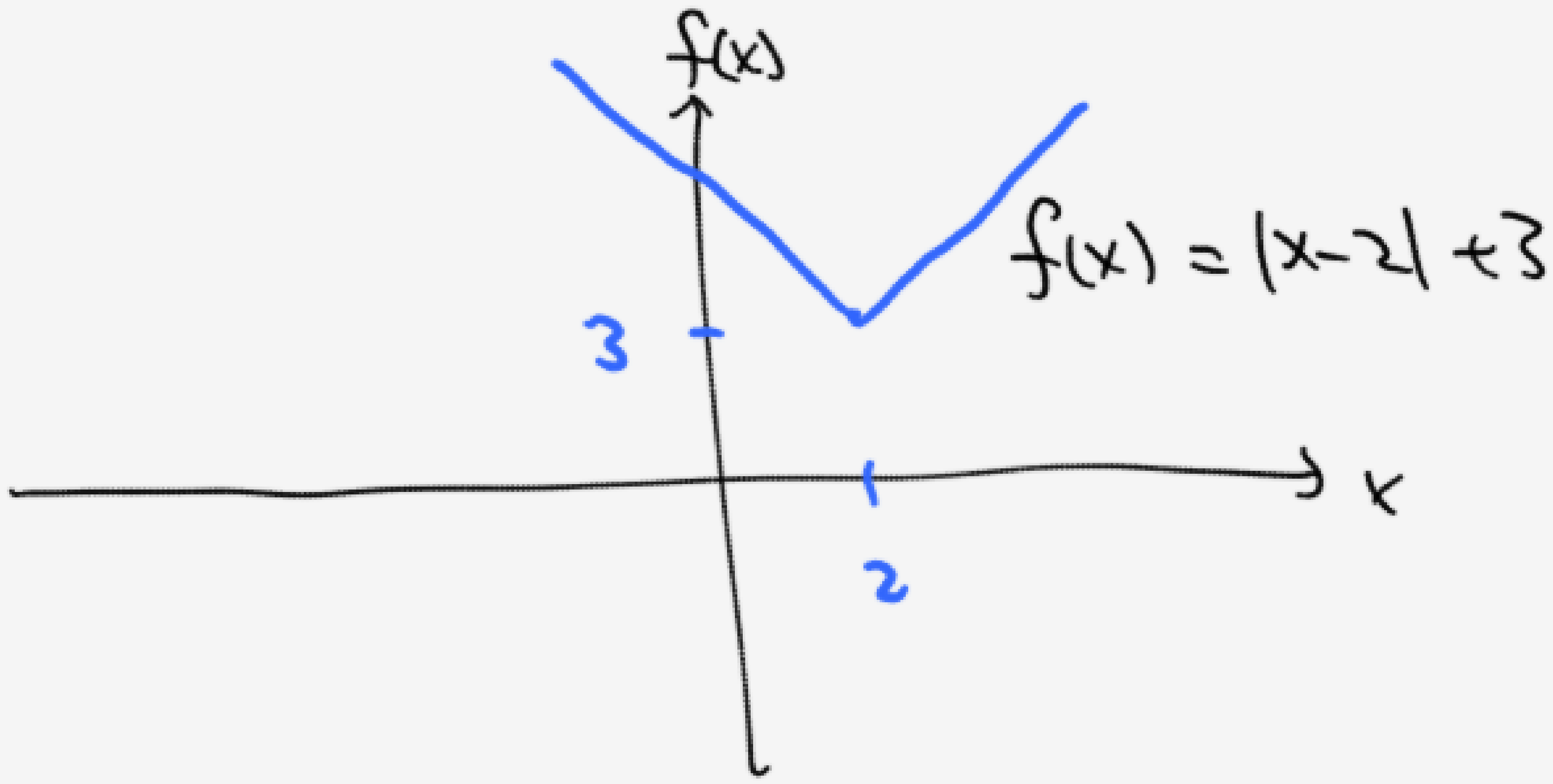
$\Rightarrow y = \dfrac{e^x - e^{-x}}{2} \Rightarrow 2ye^x = e^{2x} - 1 \Rightarrow 0 = e^{2x} - 2ye^x - 1$

$\Rightarrow e^x = \dfrac{2y \pm \sqrt{4y^2 + 4}}{2} \Rightarrow e^x = y + \sqrt{y^2+1}$ or $e^x = y - \sqrt{y^2+1} < 0$ (rej.)

$\Rightarrow x = \ln(y + \sqrt{y^2+1})$ hence f is surjective

Appendix (sketch of graphs)

2i



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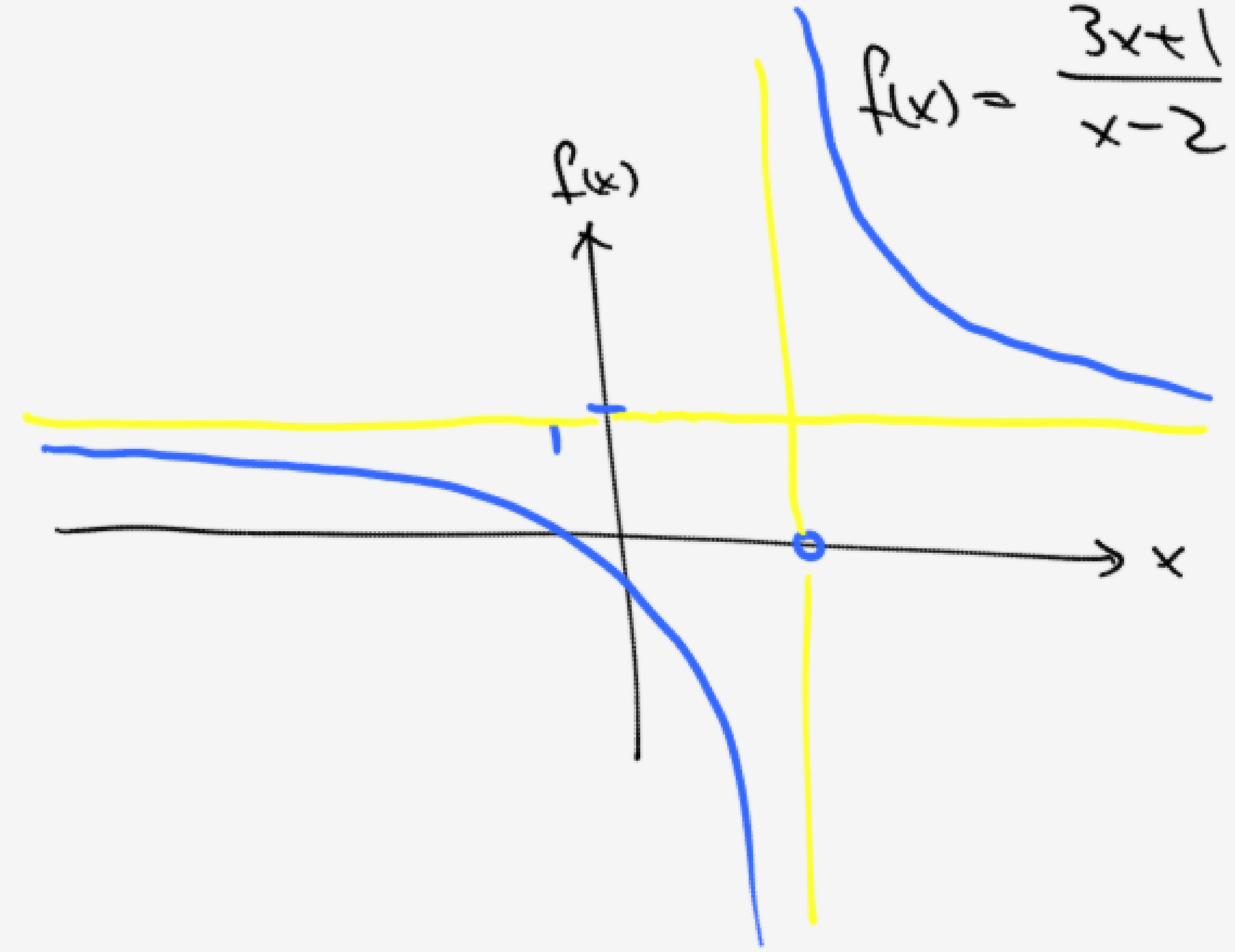

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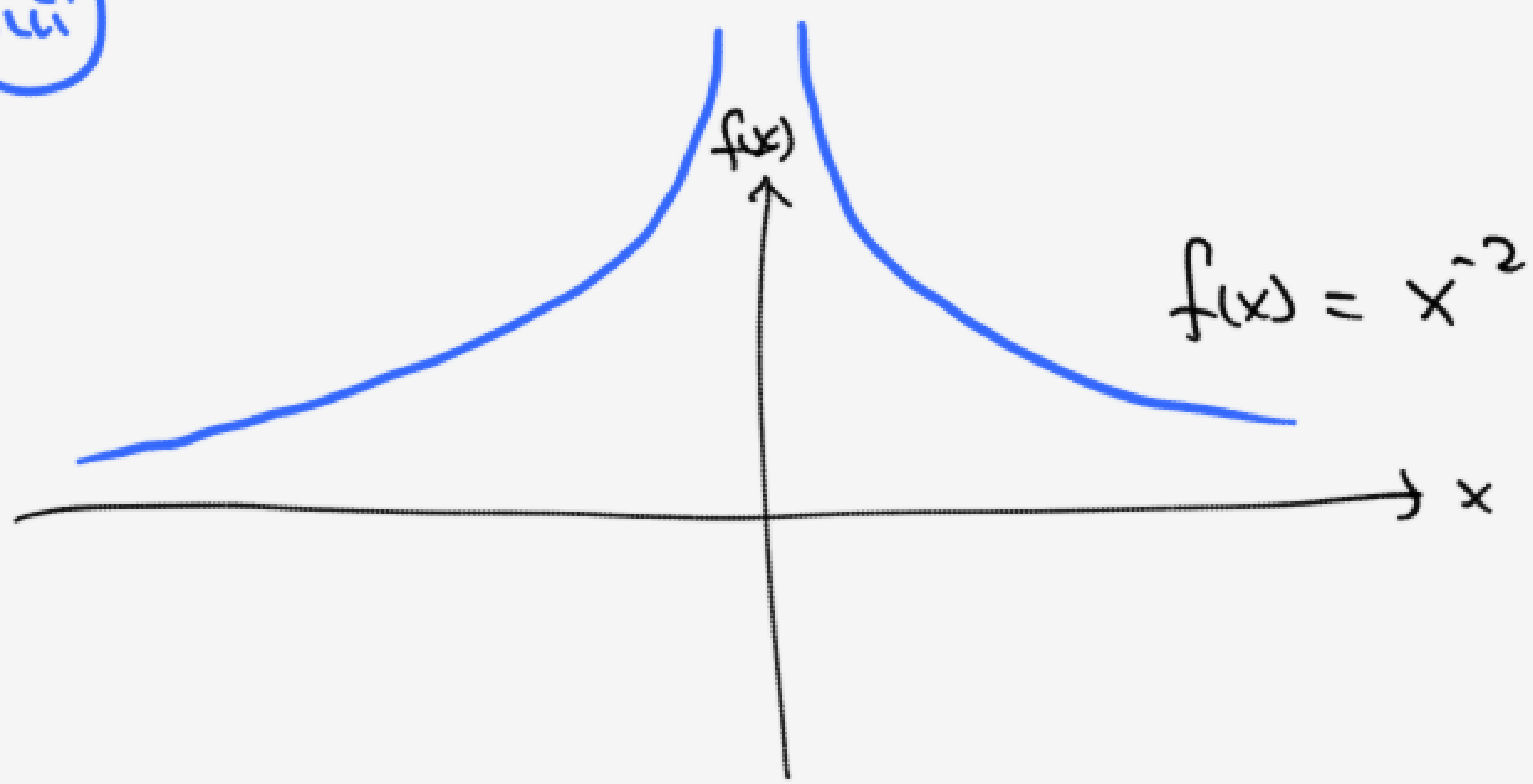

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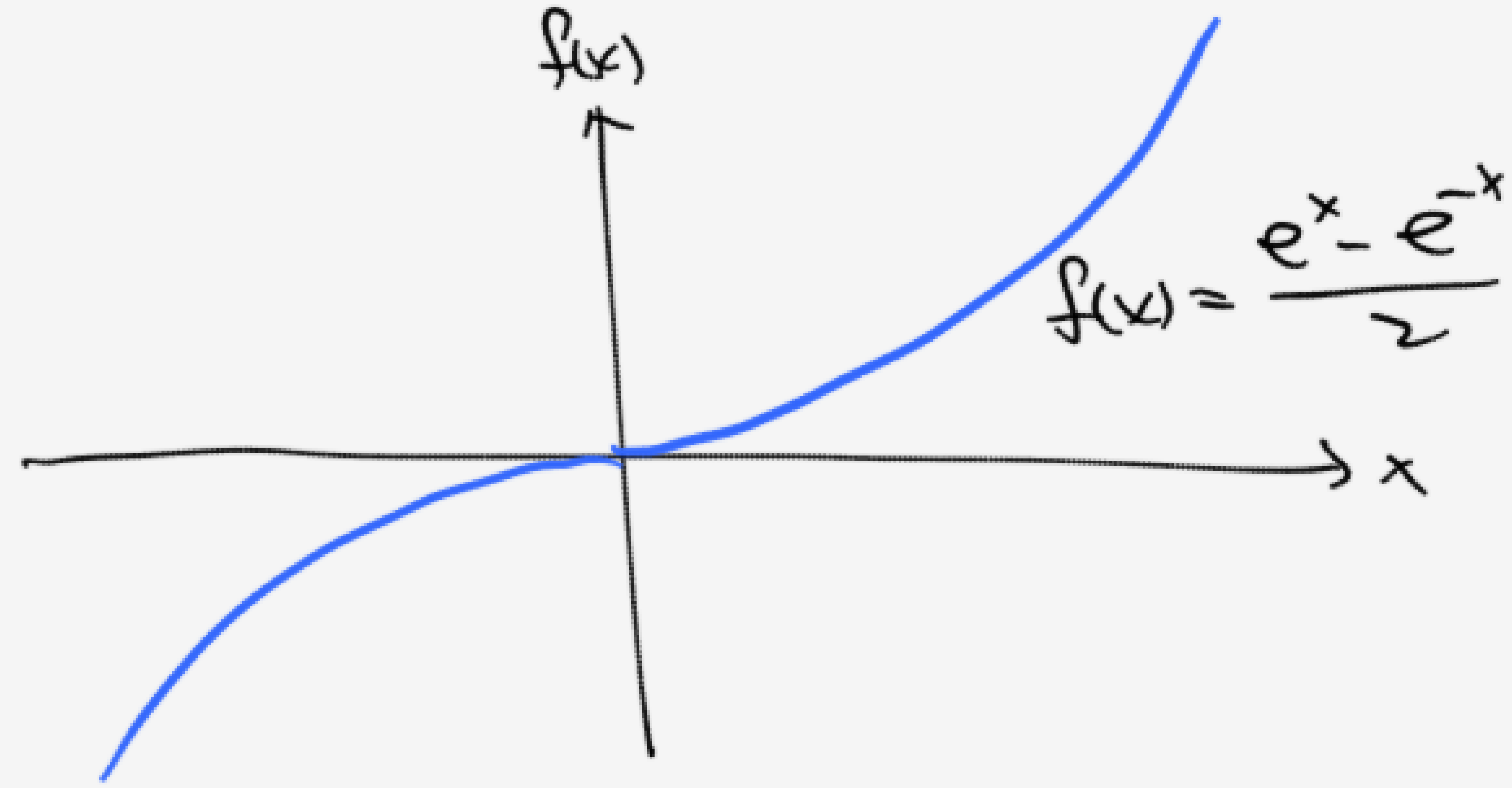
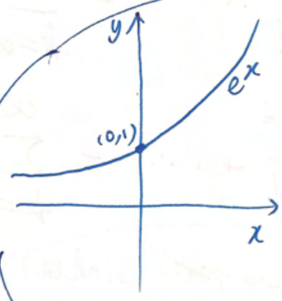


18/9/2014 Math 1010 C. Tutorial - 2.

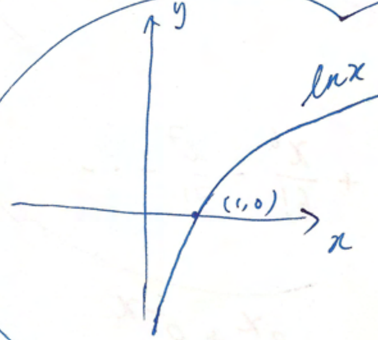
Recall:

Exponential function: $\exp(x) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$;



- monotonely increasing;
- $e^x \geq 0$;
- $\exp(x+y) = \exp(x) \cdot \exp(y)$.
- ......

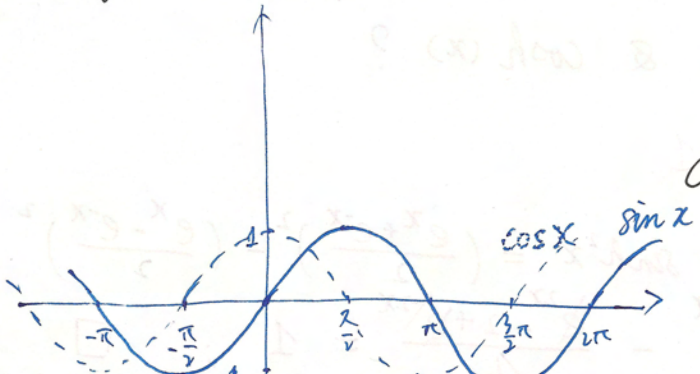
Logarithm function: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$;



- montonely increasing;
- $\ln: \mathbb{R}_{>0} \to \mathbb{R}$;
- $\ln(x \cdot y) = \ln x + \ln y$;
- $\ln x$ is the inverse fn of $\exp(x)$
- ......

Trigonometric functions: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}$;

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$;



Properties:

- **Periodic:** $\sin(x + 2\pi) = \sin x$; $\forall x \in \mathbb{R}$;
  $\cos(x + 2\pi) = \cos x$; $\forall x \in \mathbb{R}$;
- **Even & ODD:** $\sin(-x) = -\sin x$; $\forall x \in \mathbb{R}$,
  $\cos(-x) = \cos x$; $\forall x \in \mathbb{R}$;
- **Relation:** $(\sin x)^2 + (\cos x)^2 = 1$;
  (often denoted $\sin^2 x + \cos^2 x = 1$).
  $\sin(x + \frac{\pi}{2}) = \cos x$;

Now:

Hyperbolic functions:

$$\sinh(x) := x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \;;$$

l'sInt∫l

$$\cosh(x) := 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \;;$$

l'KD∫l

"odd exponent part": $\sinh(x)$

v.s $\exp(x)$?

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

"even exponent part" $\cosh(x)$

$$\exp(-x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots$$

(a) $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$; $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$;

(b) $\sinh(-x) = -\sinh(x)$; $\cosh(-x) = \cosh(x)$;

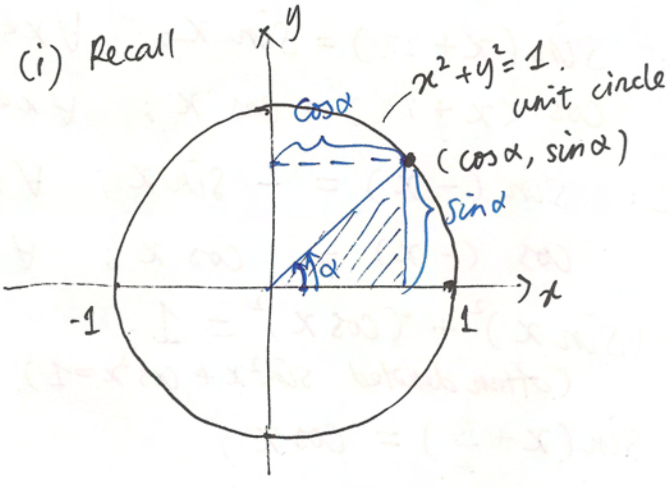
(c) Relation between $\sinh(x)$ & $\cosh(x)$?

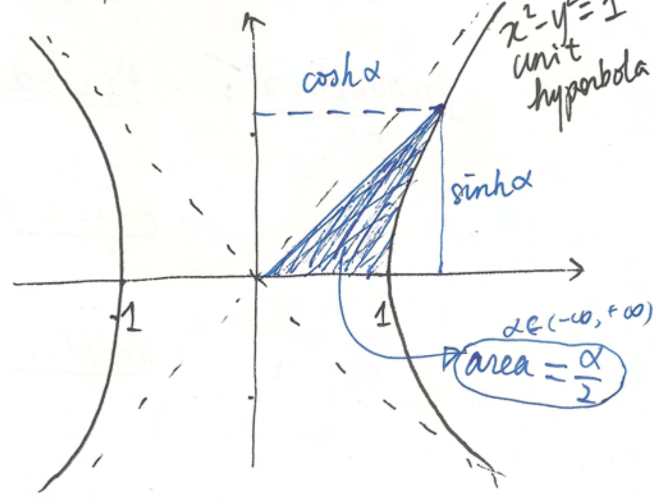
$$\cosh^2 x - \sinh^2 x = 1.$$

Reason: From (a), $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$

$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}}}{4} - \frac{\cancel{e^{2x}} - 2 + \cancel{e^{-2x}}}{4} = 1 \;; \qquad \square$$

Remark: (i) Recall





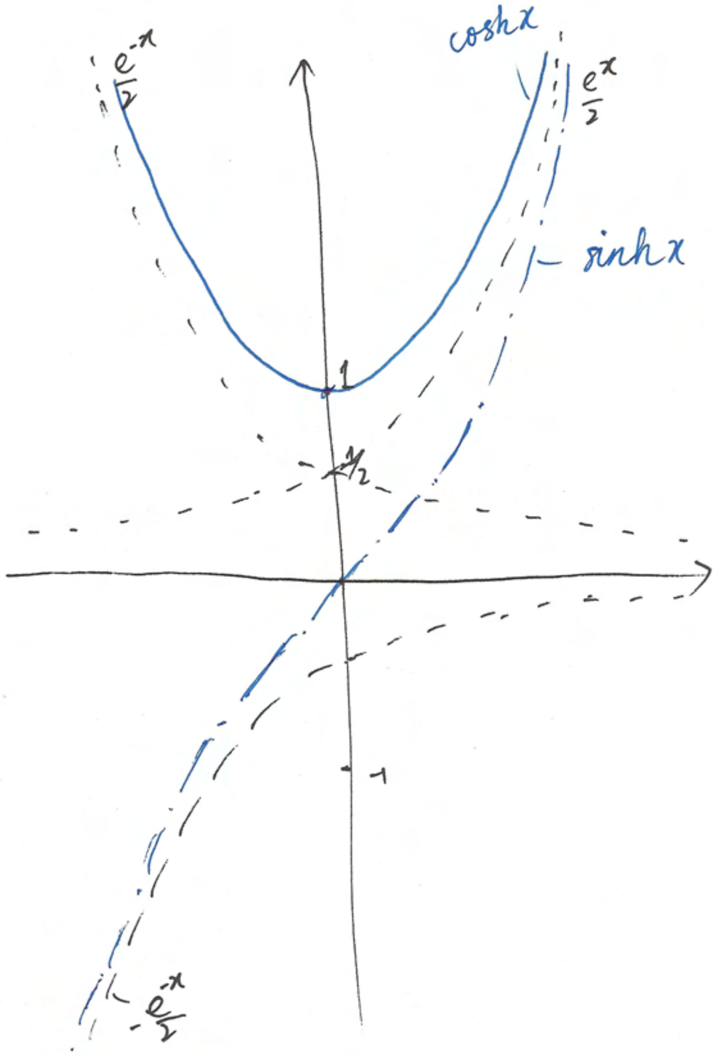
· $(\cos\alpha, \sin\alpha)$ parametrize unit ~~..~~ circle ;

· $(\cosh\alpha, \sinh\alpha)$ parametrize unit hyperbola ;

∘ 0 (" unit circle in " Minkowski space " ⟷ space-time for general ~~re~~ relativity. "

(ii) graph of $\sinh x$ & $\cosh x$;

⟶ use $\frac{1}{2}e^{x}$ & $\frac{1}{2}e^{-x}$ as auxilliary function ;



(iii) $\sinh x$ is monotonely increasing : $\sinh : \mathbb{R} \xrightarrow{\text{bijection}} \mathbb{R}$;

⇝ ∃ inverse function ; Solving $\frac{e^{x}-e^{-x}}{2} = y$;

$\Leftrightarrow (e^{x})^2 - 2y\cdot(e^{x}) - 1 = 0$. Note that $e^{x} \geq 0$

$\Rightarrow x = \ln(y + \sqrt{y^2+1})$;

Hence the inverse f'n ~~of~~ of $\sinh(x)$ is $f(x) = \ln(x + \sqrt{1+x^2})$.

($f : \mathbb{R} \xrightarrow{\text{bijection}} \mathbb{R}$).

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Math 1010C Term 1 2014
Supplementary exercises 1

The following exercises are optional, and for your own enjoyment only.

1. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *even* if $f(x) = f(-x)$ for all $x \in \mathbb{R}$, and *odd* if $f(x) = -f(-x)$ for all $x \in \mathbb{R}$.
   (a) Suppose $p\colon \mathbb{R} \to \mathbb{R}$ is the polynomial function
   $$p(x) = \sum_{n=0}^{d} a_n x^n.$$
   Show that p is even if and only if $a_n = 0$ for all odd integers n.
   (b) Let p be as in part (a). Find a necessary and sufficient condition on the coefficients of p, such that p is odd.
   (c) Is there a function $g\colon \mathbb{R} \to \mathbb{R}$ that is neither even nor odd?
   (d) Is there a function $h\colon \mathbb{R} \to \mathbb{R}$ that is both even and odd?
   (e) Show that every function $f\colon \mathbb{R} \to \mathbb{R}$ can be written as the sum of an odd function and an even function.
   (f) For those who know derivatives already: show that the derivative of an odd function is even, and the derivative of an even function is odd.
   (g) For those who know some linear algebra: Does the set of all even functions from $\mathbb{R}$ to $\mathbb{R}$ form a vector space over $\mathbb{R}$? What about the set of all odd functions?

2. The following generalizes the concept of odd and even functions defined above. Suppose X is a set, and $\theta\colon X \to X$ is an *involution*, in the sense that $\theta \circ \theta$ is the identity function on X (i.e. $\theta(\theta(x)) = x$ for all $x \in X$).
   (a) Show that $\theta\colon X \to X$ is a bijection.
   (b) A function $f\colon X \to \mathbb{R}$ is said to be even with respect to θ if $f(\theta(x)) = f(x)$ for all $x \in X$. A function $f\colon X \to \mathbb{R}$ is said to be odd with respect to θ if $f(\theta(x)) = -f(x)$ for all $x \in X$.
      (i) Find all functions $F\colon X \to \mathbb{R}$ that is both even with respect to θ, and odd with respect to θ.
      (ii) Show that every function $f\colon X \to \mathbb{R}$ can be written as the sum $g + h$, where $g\colon X \to \mathbb{R}$ is odd with respect to θ, and $h\colon X \to \mathbb{R}$ is even with respect to θ.
   (c) How is all this relevant to Question 1?
   (d) For those who know complex numbers already: Did it matter that we considered functions that took values in $\mathbb{R}$? What if we considered complex-valued functions?

3. (a) Is there a bijection from $\mathbb{N}$ to $\mathbb{Z}$? If yes, construct one.
   (b) Is there a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$? (The latter is the set of ordered pairs (m, n), where m and n are both positive integers.) If yes, construct one. (Hint: Draw a picture to visualize $\mathbb{N} \times \mathbb{N}$.)
   (c) Is there a bijection from $\mathbb{N}$ to $\mathbb{Q}$? If yes, construct one. (Hint: Use part (b).)

(d) (Challenge) A sequence of positive integers is an ordered list $(a_1, a_2, a_3, \ldots)$, where each a_i is a positive integer. The set of all sequences of positive integers is usually denoted $2^{\mathbb{N}}$. Is there a bijection from $\mathbb{N}$ to $2^{\mathbb{N}}$?

18/9/2014　　Math 1010C　　Tutorial - 2.

supplementary exercises 1.

1. Solutions:

(a). $p(x) = \sum_{n=0}^{d} a_n x^n$;

p is even $\Leftrightarrow$ $p(-x) = p(x)$, i.e. $\sum_{n=0}^{d} a_n (-x)^n = \sum_{n=0}^{d} a_n x^n$;

But $p(-x) = \sum_{n \text{ even}} a_n x^n + \sum_{n \text{ odd}} (-a_n) x^n$

Hence $p(-x) = p(x) \Leftrightarrow \sum_{n \text{ odd}} (-a_n) x^n = \sum_{n \text{ odd}} a_n x^n$ as a function of x,

$\Leftrightarrow \sum_{n \text{ odd}} (2a_n) x^n = 0$ as a function $\Leftrightarrow$ $a_n = 0, \forall n$ odd;

(b) $p(x) = \sum_{n=0}^{d} a_n x^n$; p is odd $\Leftrightarrow$ $p(-x) = -p(x)$;

But still $p(-x) = \sum_{n \text{ even}} a_n x^n + \sum_{n \text{ odd}} (-a_n) x^n$;

Hence $p(-x) = -p(x) \Leftrightarrow \sum_{n \text{ even}} a_n x^n = \sum_{n \text{ even}} (-a_n) x^n$ as a f'n of x;

$\Leftrightarrow$ $a_n = 0, \forall n$ even;

(c). Of course! $g(x) = x + 1$;

(d). $h(-x) \overset{\text{odd}}{=} -h(x)$, but $h(-x) \overset{\text{even}}{=} h(x) \Rightarrow h(x) = -h(x)$

$\Rightarrow h(x) \equiv 0$;

(e). $f(x) = \underbrace{\frac{1}{2}\left(f(x) + f(-x)\right)}_{\text{even}} + \underbrace{\frac{1}{2}\left(f(x) - f(-x)\right)}_{\text{odd}}$;

(f). $f(-x) \overset{\text{odd}}{=} -f(x) \Rightarrow f'(-x)\cdot(-1) = -f'(x) \Rightarrow f'(-x) \overset{\text{even}}{=} f'(x)$;

$f(-x) \overset{\text{even}}{=} f(x) \Rightarrow f'(-x)\cdot(-1) = f'(x) \Rightarrow f'(-x) \overset{\text{odd}}{=} -f'(x)$;

(g). YES, THEY ARE ALL VECTOR SPACE.

$\square$

2. (a) $\theta: X \to X$ involution, i.e. $\theta(\theta(x)) = x$, $\forall x$;

then θ is surjective, since $\forall y \in X$, $\exists x := \theta(y)$, s.t $\theta(x) = \theta(\theta(y)) = y$;

θ is injective, since if $x_1, x_2 \in X$, s.t $\theta(x_1) = \theta(x_2)$, then apply $\theta\circ$ too both sides

$x_1 = \theta(\theta(x_1)) = \theta(\theta(x_2)) = x_2$;

(b). (i) $\forall x \in X$, $F(x) = F(\theta(x)) = -F(x) \Rightarrow F(x) \equiv 0$;

(ii) $f(x) = \underbrace{\frac{1}{2}(f(x) - f(\theta(x)))}_{g} + \underbrace{\frac{1}{2}(f(x) + f(\theta(x)))}_{h}$

$g(\theta(x)) = -g(x)$, $\qquad h(\theta(x)) = h(x)$.

(c). Em?

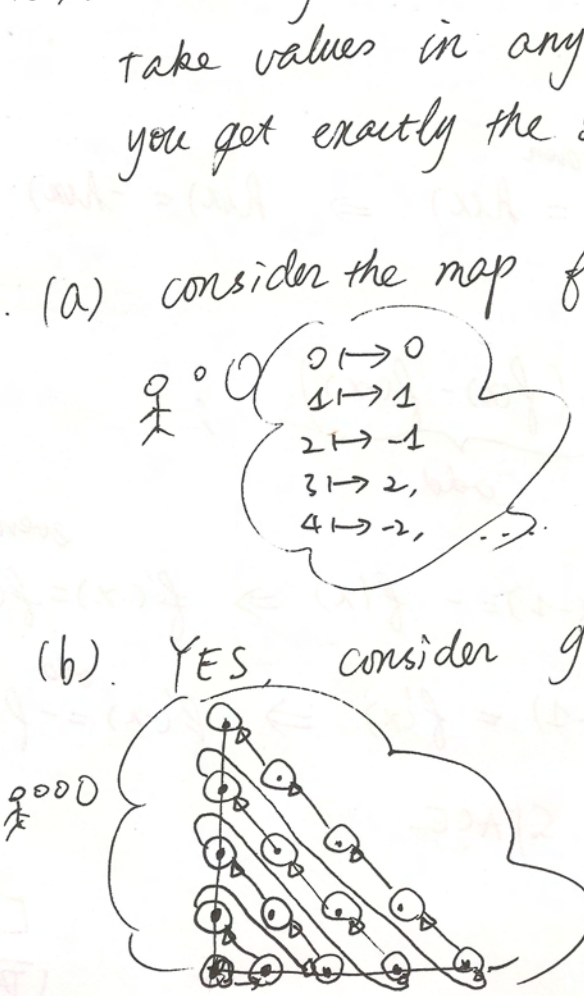
(d). Exactly the same. Actually you can consider f'ns take values in any field $\mathbb{k}$ s.t $char(\mathbb{k}) \neq 2$, then you get exactly the same story.

3. (a) consider the map $f: \mathbb{N} \to \mathbb{Z}$, s.t. $\begin{cases} f(2n) = -n; \\ f(2n+1) = n+1; \end{cases}$

$0 \mapsto 0$
$1 \mapsto 1$
$2 \mapsto -1$
$3 \mapsto 2,$
$4 \mapsto -2,$ ...

f is bijection of $\mathbb{N}$ and $\mathbb{Z}$.

(b). YES, consider $g: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, s.t



$$g(n) = \left(N - \left(n - \frac{N(N+1)}{2} - 1\right), \; n - \frac{N(N+1)}{2} - 1 \right)$$

for $N \in \mathbb{N}$ s.t. $\frac{N(N+1)}{2} < n \leq \frac{(N+1)(N+2)}{2}$;

(c). First, construct a bijection from $\mathbb{Q}_+ := \{ \frac{p}{q} \in \mathbb{Q} \mid \frac{p}{q} > 0 \}$ to $\mathbb{N} \times \mathbb{N}$, indicated as follows:

$$\mathbb{N}\times\mathbb{N} \xrightarrow[\sim]{h} \mathbb{Q}_+$$

$$\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\
\frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} & \frac{17}{2} & \frac{19}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{4}{3} & \frac{5}{3} & \frac{7}{3} & \frac{8}{3} & \frac{10}{3} & \frac{11}{3} & \frac{13}{3} & \frac{14}{3} & \cdots \\
\frac{1}{4} & \frac{3}{4} & \frac{5}{4} & \frac{7}{4} & \frac{9}{4} & \frac{11}{4} & \frac{13}{4} & \frac{15}{4} & \frac{17}{4} & \frac{19}{4} & \cdots \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{6}{5} & \frac{7}{5} & \frac{8}{5} & \frac{9}{5} & \frac{11}{5} & \frac{12}{5} & \cdots \\
\frac{1}{6} & \frac{5}{6} & \frac{7}{6} & \frac{11}{6} & \frac{13}{6} & \frac{17}{6} & \frac{19}{6} & \frac{23}{6} & \frac{25}{6} & \frac{29}{6} & \cdots \\
\frac{1}{7} & \frac{2}{7} & \frac{3}{7} & \frac{4}{7} & \frac{5}{7} & \frac{6}{7} & \frac{8}{7} & \frac{9}{7} & \frac{10}{7} & \frac{11}{7} & \cdots \\
\frac{1}{8} & \frac{3}{8} & \frac{5}{8} & \frac{7}{8} & \frac{9}{8} & \frac{11}{8} & \frac{13}{8} & \frac{15}{8} & \frac{17}{8} & \frac{19}{8} & \cdots \\
\frac{1}{9} & \frac{2}{9} & \frac{4}{9} & \frac{5}{9} & \frac{7}{9} & \frac{8}{9} & \frac{10}{9} & \frac{11}{9} & \frac{13}{9} & \frac{14}{9} & \cdots \\
\frac{1}{10} & \frac{3}{10} & \frac{7}{10} & \frac{9}{10} & \frac{11}{10} & \frac{13}{10} & \frac{17}{10} & \frac{19}{10} & \frac{21}{10} & \frac{23}{10} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & & & &
\end{array}$$

then compose with $\mathbb{N} \xrightarrow{g} \mathbb{N} \times \mathbb{N} \xrightarrow[\sim]{h} \mathbb{Q}_+$ gives $\mathbb{N} \xrightarrow[\sim]{\text{bijection}} \mathbb{Q}_+$, $\ell := h(g)$

extends to $\mathbb{Z} \xrightarrow[\mathcal{L}]{\sim} \mathbb{Q}$ by $\begin{cases} \mathcal{L}(n) = \ell(n+1), & n \geq 0; \\ \mathcal{L}(-n) = -\ell(n), & n < 0; \end{cases}$

but $\mathbb{N} \xrightarrow[\sim]{f} \mathbb{Z}$ bijection, hence

$\mathbb{N} \xrightarrow{f} \mathbb{Z} \xrightarrow[\sim]{\mathcal{L}} \mathbb{Q}$ is bijection. $\square$

Rmk: One can construct directly bijection from $\mathbb{N}$ to $\mathbb{Q}$ as follows:

$\mathbb{N}$

$\downarrow$

$\mathbb{Q}$

$$
\begin{array}{cccccccccc}
0 & 1 & (-1) & 2 & (-2) & 3 & (-3) & 4 & (-4) & 5 \ \ (-5) \ \cdots \\
\frac{1}{2} & (-\frac{1}{2}) & \frac{3}{2} & (-\frac{3}{2}) & \frac{5}{2} & (-\frac{5}{2}) & \frac{7}{2} & (-\frac{7}{2}) & \cdots & \\
\frac{1}{3} & (-\frac{1}{3}) & \frac{2}{3} & (-\frac{2}{3}) & \frac{4}{3} & (-\frac{4}{3}) & \cdots & & & \\
\frac{1}{4} & (-\frac{1}{4}) & \frac{3}{4} & (-\frac{3}{4}) & \cdots & & & & &
\end{array}
$$

2^{∞} $\{0,1\}^{\mathbb{N}} = \{\text{maps from } \mathbb{N} \text{ to } \{0,1\}\}$

$= \{\text{sequence } \underline{a} = (a_1, a_2, \cdots) \text{ s.t } a_i \in \{0,1\}\}$

(d). Is there a bijection from $\mathbb{N}$ to $2^{\mathbb{N}}$?

Prove by contradiction. Assume YES, then we can list all elements of $2^{\mathbb{N}}$ in a sequence, say $\{\underline{a}_n\}_{n=0}^{\infty}$; where $\underline{a}_n$ is a sequence $(a_{n1}, a_{n2}, \cdots\cdots)$. We list all elts of $2^{\mathbb{N}}$ as follows: $a_{ij} \in \{0,1\}$

$\underline{a}_1$: $(a_{11}, a_{12}, a_{13}, a_{14}, \cdots\cdots\cdots)$ $\quad a_{1j} \in \{0,1\}$;

$\underline{a}_2$: $(a_{21}, a_{22}, a_{23}, a_{24}, \cdots\cdots\cdots)$ $\quad a_{2j} \in \{0,1\}$;

$\underline{a}_3$: $(a_{31}, a_{32}, a_{33}, a_{34}, \cdots\cdots\cdots)$ $\quad a_{3j} \in \{0,1\}$;

$\vdots \qquad\qquad \vdots$

Now consider a sequence $\underline{x} = (x_1, x_2, x_3, \cdots)$ $\quad x_j \in \{0,1\}$.

s.t $\underline{x_n = a_{nn} + 1}$ (mod 2). (The main point is: $x_n \neq a_{nn}$, $\forall n \in \mathbb{N}$)

Then $\underline{x}$ can not appear in above list,

Because, if, say. $\underline{x} = \underline{a}_N$ for some $N \in \mathbb{N}$, then

$(x_1, x_2, \cdots, x_N, \cdots)$ $\quad (a_{N1}, a_{N2}, a_{N3}, \cdots, a_{NN}, a_{N(N+1)}, \cdots)$

$\underline{x_N = a_{NN}}$ but $x_N = a_{NN} + 1$ By construction.

Contradiction! $\square$

MATH 1010C. TUTORIAL-2. SUPPLEMENT: 18/9/2014

Q: WHY trigonometric functions are *periodic* ?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$$

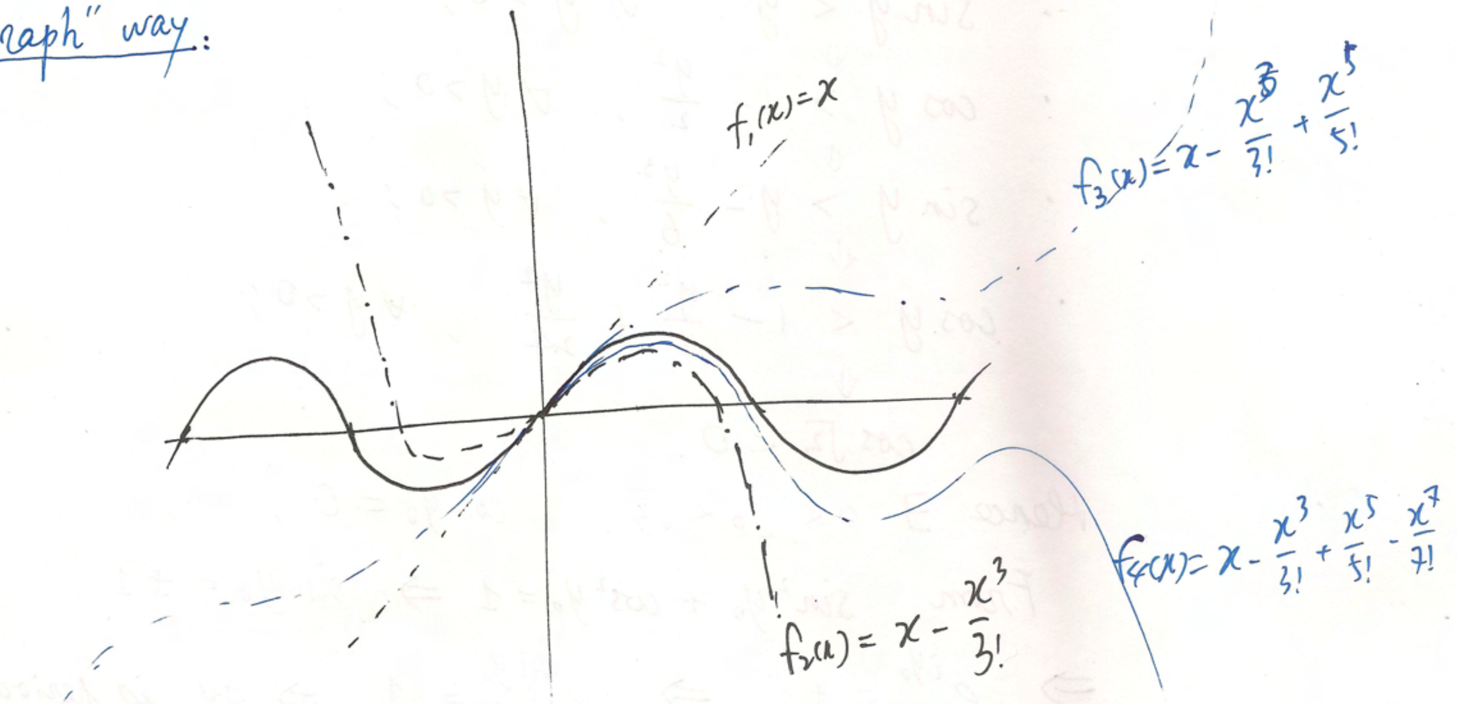
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots$$

Quick way:
$$\frac{\pi^2}{\sin^2 \pi x} = \sum_{n=-\infty}^{+\infty} \frac{1}{(x-n)^2} = \frac{1}{x^2} + \frac{1}{(x-1)^2} + \frac{1}{(x+1)^2} + \frac{1}{(x-2)^2} + \cdots\cdots$$

$\rightsquigarrow$ $\sin^2 x$ has period π $\rightsquigarrow$ $\cos x = 1 - 2\sin^2 \frac{x}{2}$ has period 2π $\rightsquigarrow$ $\sin x = \cos(x - \frac{\pi}{2})$ has period 2π.

Problem: "cheating". What is π ? & WHY $\sin x = \cos(x - \frac{\pi}{2})$?

"Graph" way:



EX: Use Mathematica OR Matlab OR ... TO DRAW PICTURES of $f_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}$ to convince yourself!

"Honest" (& CORRECT) WAY: (Use "derivatives" & complex number $i := \sqrt{-1}$;)

$i^2 = -1$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} + \cdots$$
$$= \cos x + i \sin x;$$

TA2-supple

Hence

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \; ; \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \; ;$$

POINT: We show that function e^{ix} has smallest period $0 < 2\pi < 7$ ($< 4\sqrt{3}$) ; $\rightsquigarrow$ $\sin x$ & $\cos x$ has period 2π.

REASON: We want to use $e^{x+y} = e^x \cdot e^y$;

since $e^{i(x+\omega_0)} = e^{ix} \cdot e^{i\omega_0}$, $\omega_0 > 0$;

We want smallest positive $\omega_0 > 0$, s.t. $e^{i\omega_0} = 1$.

Method: • From theory of derivatives, one can show:

- $\sin y < y$, $\forall y > 0$;
$\Downarrow$
- $\cos y > 1 - \frac{y^2}{2}$, $\forall y > 0$;
$\Downarrow$
- $\sin y > y - \frac{y^3}{6}$, $\forall y > 0$;
$\Downarrow$
- $\cos y < 1 - \frac{y^2}{2} + \frac{y^4}{24}$, $\forall y > 0$;
$\Downarrow$

$\cos\sqrt{3} < 0$.

Hence $\exists \; 0 < y_0 < \sqrt{3}$, $\cos y_0 = 0$;

From $\sin^2 y_0 + \cos^2 y_0 = 1 \Rightarrow \sin y_0 = \pm 1$;

$\Rightarrow e^{iy_0} = \pm i \Rightarrow e^{4iy_0} = 1. \Rightarrow 4y_0$ is period.

• $4y_0$ is the smallest period of e^{ix}:

$\forall \; 0 < y < y_0$. Then $\sin y > y(1 - \frac{y^2}{6}) > \frac{y}{2} > 0$;

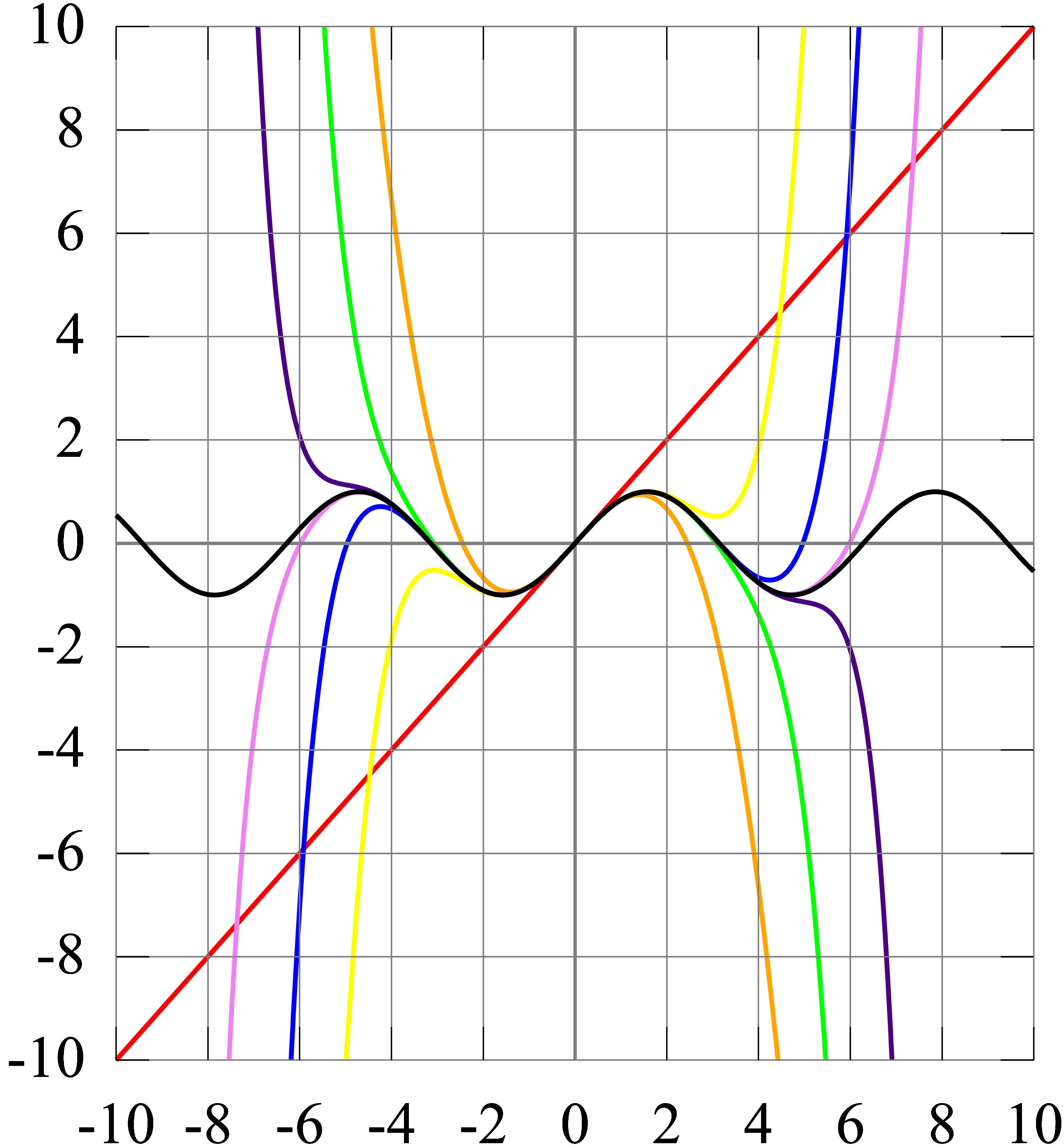
$\Rightarrow \cos y$ is strictly decreasing on $0 < y < y_0$, &

$\Rightarrow \sin y$ is strictly increasing on $0 < y < y_0$;

$\Rightarrow \forall \; 0 < y < y_0$. $0 < \sin y < 1 \Rightarrow e^{iy} \neq \pm 1$, or $\pm i$;

$\Rightarrow e^{4iy} \neq 1. \Rightarrow 4y_0$ is the smallest positive period.

Define $\pi := 2y_0$, half of the period of e^{ix}. Then period is 2π. $\square$

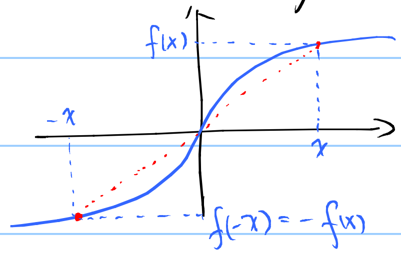

10
8
6
4
2
0
-2
-4
-6
-8
-10
-10 -8 -6 -4 -2 0 2 4 6 8 10

Tutorial 2, Min, Jie jmin@math.cuhk.edu.hk

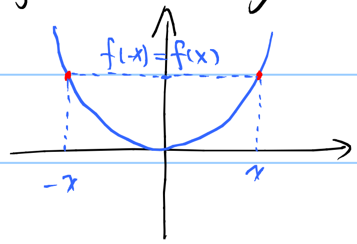
便笺标题 2014/9/15

Some properties of functions

* odd : $f(-x) = -f(x)$, symmetric with respect to the origin.



even: $f(-x) = f(x)$, symmetric with respect to y-axis



decide whether the function is even or odd:

(1) $f(x) = x^3$

$f(-x) = (-x)^3 = -x^3 = -f(x)$

So f is odd

(2) $f(x) = \frac{1}{x-1}$

$f(-x) = \frac{1}{-x-1}$

Take $x=2$ for example, $f(2) = 1$, $f(-2) = -\frac{1}{3}$

So f is neither odd nor even.

Also exist function both odd and even.

* Injective : If $f(x_1)=f(x_2)$, then $x_1=x_2$. $(\forall x_1, x_2 \in \text{domain})$

Surjective : $\forall y \in$ codomain, can find $x \in$ domain s.t. $y=f(x)$

Solve Equations :

Eg. (1) $f: \mathbb{R}\setminus\{-3\} \to \mathbb{R}$, $f(x)=\dfrac{2x-9}{x+3}$

Injective ? : if $0=f(x_1)-f(x_2)=\dfrac{2x_1-9}{x_1+3}-\dfrac{2x_2-9}{x_2+3}$

$$=\frac{(2x_1-9)(x_2+3)-(2x_2-9)(x_1+3)}{(x_1+3)(x_2+3)}$$

$$\Leftrightarrow \quad 2x_1x_2-9x_2+6x_1-27-(2x_1x_2-9x_1+6x_2-27)=0$$

$$\Leftrightarrow \quad 15(x_1-x_2)=0$$

$$\Leftrightarrow \quad x_1=x_2$$

So $f(x)=\dfrac{2x-9}{x+3}$ is injective.

Surjective ? : $\forall y \in \mathbb{R}$, if $y=\dfrac{2x-9}{x+3}$

$\Leftrightarrow \quad xy+3y=2x-9 \quad \Leftrightarrow \quad x=\dfrac{3y+9}{2-y}$

So we have no sol'n to x when $y=2$

So f is not surjective.

(2) $f: \mathbb{R}\to\mathbb{R}$, $f(x)=\ln(x+\sqrt{x^2+1})$

Injective ? : if $0=f(x_1)-f(x_2)=\ln(x_1+\sqrt{x_1^2+1})-\ln(x_2+\sqrt{x_2^2+1})$

$$=\ln\left(\frac{x_1+\sqrt{x_1^2+1}}{x_2+\sqrt{x_2^2+1}}\right)$$

$$\Leftrightarrow \quad 1=\frac{x_1+\sqrt{x_1^2+1}}{x_2+\sqrt{x_2^2+1}}$$

$$\Leftrightarrow \quad (x_1 - x_2) + \sqrt{x_1^2+1} - \sqrt{x_2^2+1} = 0$$

$$\Leftrightarrow \quad (x_1 - x_2) + \frac{x_1^2 - x_2^2}{\sqrt{x_1^2+1} + \sqrt{x_2^2+1}} = 0$$

$$\Leftrightarrow \quad (x_1 - x_2)\left(1 + \frac{x_1 + x_2}{\sqrt{x_1^2+1} + \sqrt{x_2^2+1}}\right) = 0$$

i.e. at least one of the two has to be 0

Since $|x_1 + x_2| \leq |x_1| + |x_2|$

$$\sqrt{x_1^2+1} + \sqrt{x_2^2+1} > |x_1| + |x_2|$$

So $\left|\frac{x_1 + x_2}{\sqrt{x_1^2+1} + \sqrt{x_2^2+1}}\right| < 1$

So $-1 < \frac{x_1 + x_2}{\sqrt{x_1^2+1} + \sqrt{x_2^2+1}} < 1$

So $1 + \frac{x_1 + x_2}{\sqrt{x_1^2+1} + \sqrt{x_2^2+1}} > 0$, i.e. $\neq 0$

So $x_1 - x_2 = 0$

f is injective

Surjective? $\forall y \in \mathbb{R}$, $y = \ln(x + \sqrt{x^2+1})$

$$\Leftrightarrow \quad e^y = x + \sqrt{x^2+1}$$

$$(e^y - x)^2 = x^2 + 1$$

$$x^2 - 2xe^y + e^{2y} = x^2 + 1$$

$$x = \frac{e^{2y} - 1}{2e^y}$$

i.e. $\forall y \in \mathbb{R}$, we can find $x = \frac{e^{2y}-1}{2e^y}$ s.t. $f(x) = y$.

so f is surjective.

Remark: Strictly increasing / decreasing $\Rightarrow$ injective

* Exercise: determine whether the function is injective, surjective or not.

(1) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

(2) $f: \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = |x-2| + 3$

(1) Injective: $0 = f(x_1) - f(x_2) = \dfrac{x_1}{\sqrt{x_1^2+1}} - \dfrac{x_2}{\sqrt{x_2^2+1}} = \dfrac{x_1\sqrt{x_2^2+1} - x_2\sqrt{x_1^2+1}}{\sqrt{x_1^2+1}\sqrt{x_2^2+1}}$

$\Leftrightarrow \quad x_1\sqrt{x_2^2+1} - x_2\sqrt{x_1^2+1} = 0$

$$\frac{x_1^2(x_2^2+1) - x_2^2(x_1^2+1)}{x_1\sqrt{x_2^2+1} + x_2\sqrt{x_1^2+1}} = 0$$

$\Leftrightarrow \quad x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2) = 0$

Note $f(x_1) \neq f(x_2)$ when $x_1 = -x_2$, so we reject "$x_1 + x_2 = 0$"

so $x_1 = x_2$.

Surjective: $\forall y \in \mathbb{R}$, $y = \dfrac{x}{\sqrt{x^2+1}}$

$y^2 = \dfrac{x^2}{x^2+1}$, $\Leftrightarrow$ $x^2y^2 + y^2 = x^2$ $\Leftrightarrow$ $x^2 = \dfrac{y^2}{1-y^2}$

So when $y = \pm 1$, there is no sol'n for x.

(2) $f(x) = |x-2| + 3$.

Injective: $0 = f(x_1) - f(x_2) = |x_1 - 2| - |x_2 - 2|$

$\Leftrightarrow \quad |x_1 - 2| = |x_2 - 2| \quad \Leftrightarrow \quad (x_1-2)^2 = (x_2-2)^2$

$\Leftrightarrow \quad (x_1 - x_2)(x_1 + x_2 - 4) = 0$

$x_1 = x_2$ or $x_1 + x_2 = 4$. NOT injective

Surjective: $\forall\ y \in \mathbb{R}^+$, $y = |x-2| + 3$

$|x-2| = y-3 \geqslant 0$

So we have sol'n for x only if $y \geqslant 3$.

NOT surjective.